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A GROTHENDIECK FACTORIZATION THEOREM ON 2-CONVEX SCHATTEN SPACES

BY

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ABSTRACT

We prove that for every bounded linear operator $T: C^{2p} \to H$ ($1 \leq p < \infty$, H is a Hilbert space, C^{2p} is the Schatten space) there exists a continuous linear form f on C^p such that $f \geq 0$, $\| f \|_{(C^p)^*} = 1$ and

$$\forall x \in C^{2p}, \quad \| T(x) \| \leq 2\sqrt{2} \| T \| < f, \frac{x^*x + xx^*}{2} >^{1/2} .$$

For $p = \infty$ this non-commutative analogue of Grothendieck's theorem was first proved by G. Pisier. In the above statement the Schatten space C^{2p} can be replaced by $C_{E^{(2)}}$ where $E^{(2)}$ is the 2-convexification of the symmetric sequence space E, and f is a continuous linear form on C_E. The statement can also be extended to $L_{E^{(2)}}(M, \tau)$ where M is a Von Neumann algebra, τ a trace on M, E a symmetric function space.

Introduction

This work fits in a long series of papers extending properties of symmetric sequence spaces E to Schatten (unitary) spaces C_E, and properties of symmetric function spaces on $[0,1]$ or $]0,\infty[$ to $L_E(M,\tau)$ spaces defined on a Von Neumann algebra M and a suitable trace τ: see for example [GTJ], [TJ], [AL], [A], [FK], [X1], [X2], [X3]. We first study the set $\mathcal{S}_A$ of norm one linear functionals supporting the unit ball of C_E at A (part II). We think that these results are interesting in their own right, we use them later on.

Our main aim is to prove a factorization theorem which extends Grothendieck's theorem for $C(K)$ (continuous functions on the compact set K) [cf P2, theorem 5.4] and Pisier's theorem for C^*-algebras [P1] :

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THEOREM 0: *Let $\mathcal{H}$ be a Hilbert space, E a symmetric sequence space, $E^{(2)}$ its 2-convexification. We assume that E^* has a strictly increasing norm. Let T be a bounded linear operator : $C_{E^{(2)}} \to \mathcal{H}$. Then there exists a positive linear form f on C_E, $\| f \| = 1$, such that*

$$\forall x \in C_{E^{(2)}} \qquad \| T(x) \| \leq 2\sqrt{2} \| T \| < f, \frac{x^*x + xx^*}{2} >^{1/2} .$$

We recall that if $C(K)$ replaces $C_{E^{(2)}}$ the constant is less than $\sqrt{\frac{\pi}{2}}$ and the constant for a C^*-algebra is less than 2.

We will also prove a generalization of Theorem 0 for $L_{E^{(2)}}(M, \tau)$ spaces (see Theorem V.5).

Theorem 0 has been predicted for many years. The commutative version, for operators: $E^{(2)} \to \mathcal{H}$ where $E^{(2)}$ is a 2-convex Köthe function space, is Grothendieck's theorem for $E = E^{(2)} = C(K)$; the general case is proved in [M, proof of Theorem 28] as a consequence of the $C(K)$ case, modulo an analogue of [P1, Proposition 1.1].

The case C_{c_0} ($= K(H)$, compact operators on a Hilbert space H) is contained in Pisier's theorem. The case $C^{2p} = C_{\ell^{2p}} = C_{(\ell^p)^{(2)}}$ ($2 \leq p < \infty$) was already known: actually it was proved in [LPP, Theorem IV.4] that Theorem 0 holds true for $C_{E^{(2)}}$ if and only if Khintchine inequalities hold true in the dual space of $C_{E^{(2)}}$; these inequalities hold true in the dual space of C^{2p} for $1 \leq p \leq \infty$ [LPP, Corollary III.4].

This was our motivation and our knowledge when we first tried to prove Theorem 0. In a previous version of this paper we gave a proof of Theorem 0 which did not extend completely to the $L_{E^{(2)}}(M, \tau)$ setting. After we had submitted this first version we learnt from the referee a proof for C^{2p} which he had known for several years but which had never been published. Some steps were similar to ours, some fitted only in the C^{2p} case. Let us now make some comments on the proof we present here.

Besides [P1] and [LPP] other proofs of Pisier's theorem for C^*-algebras can be found in [Kai, Theorem 2] and [H, Appendix]. Our proof (as well as the referee's) follows Haagerup's line. Haagerup uses the easy equality

$$\| Id + itB - \frac{t^2}{2} B^2 \| = \| e^{itB} \| + o(t^2) = 1 + o(t^2)$$

for hermitian B, $t \in \mathbb{R}$, in the operator norm.

In our case the idea is to majorize $\| A+itB \|^2_{C_{E(2)}}$, up to the order 2, around $t=0$, for $A \geq 0$ and B hermitian (the referee's proof uses $E \| A+\epsilon tB \|^p_{C_p}$ instead, where ϵ is a random variable such that $P(\epsilon=+1)=P(\epsilon=-1)=1/2$). In the finite dimensional case we get

$$(1) \qquad \| A+itB \|^2_{C_{E(2)}} \leq \| A^2+2t^2B^2 \|_{C_E} + o(t^2).$$

This easily implies theorem 0 if C_E is smooth and if $T\colon C_{E(2)} \to \mathcal{H}$ attains its norm at $A \geq 0$, $A \in C_{E(2)}$ (see Lemma IV.3); the desired linear form is the tangent linear functional at A^2. Note that for $1<p<\infty$, C^p is smooth and $\mathcal{S}_A=\{A^{p-1}\}$ for $A \geq 0$, $\| A \|_{C^p}=1$. This makes everything simpler, in particular all the results of part II are obvious in this case. If A^2 is not a smooth point for C_E we consider $\| A+itB-t^2C \|_{C_{E(2)}}$ for $A \geq 0$, B hermitian, C depending on A, B and we have to "choose" a linear form in $\mathcal{S}_{A^2}$.

The difference between our two versions (and between the referee's proof) lies in the way these norms are estimated. In our first version we used an argument of perturbation theory in finite dimensional spaces, namely order 2 expansions of the eigenvalues of $| A+itB |^2$ around $t=0$ (see the comments on part III). Our argument now relies on the fact that for suitable B's the estimation of the norm is easy and the set of these B's is big enough (see part III).

Let us mention a related norm inequality [TJ, Proposition 1]: Let $A \geq 0$, B be hermitian operators in $C_{E(2)}$. Then

$$(2) \qquad \| A^2 \|_{C_E} \leq \| A+itB \|^2_{C_{E(2)}} \leq \| A^2 \|_{C_E} + 2t^2 \| B^2 \|_{C_E}.$$

The paper is organized as follows: in part I we give notation and definitions for C_E spaces; in part II we study the set $\mathcal{S}_A$ of supporting linear functionals at $A \in C_E$ and the expansion of $\| A+tB \|_{C_E}$ around $t=0$ up to the order 1; in part III we study $\| A+itB \|_{C_{E(2)}}$ around $t=0$ up to the order 2 for some hermitian A's and B's; in part IV we prove Theorem 0, first in the finite dimensional case, then we give the reduction steps from the general case to the finite dimensional one; part V is devoted to the $L_E(M,\tau)$ setting: we give definitions and we generalize results from parts II, III, IV in order to prove Theorem V.5 which is the version of Theorem 0 in this setting.

We choose to consider separately the cases C_E and $L_E(M,\tau)$ because the first one is simpler and familiar to more readers, and the proof is more transparent.

We could have simplified the statement of some lemmas in parts II, III, IV if our aim had been only the proof of Theorem 0, because most results are needed in the finite dimensional case only; but some are of interest in the infinite dimensional case, moreover we also wanted to prepare the extension to the $L_E(M,\tau)$ setting.

I. Notation and definitions (C_E spaces)

All Banach spaces in this paper are complex Banach spaces. The dual of a Banach space X is denoted by X^*.

Definition I.1: A Banach lattice E has a strictly increasing norm if for $x, y \geq 0$, $x, y \in E$, $\| x+y \| = \| x \|$ implies $y = 0$.

We will use this property only when $x \wedge y = 0$. For example, the norm of E is strictly increasing if E is the dual space of a smooth space, or more generally if E is q-concave with constant 1

For a Banach lattice E and $1 < p < \infty$, $E^{(p)}$ denotes the p-convexification of E, i.e. $\| x \|^p_{E^{(p)}} = \| |x|^p \|_E$ [LT, 1d, p. 53]. For example $(\ell^p)^{(2)} = \ell^{2p}$, $1 \leq p \leq \infty$.

A symmetric sequence space (see [S] and [LT, Definition 2.a.1]) is a Banach lattice of bounded complex sequences equipped with a norm such that

(i) $\| (a_0, \ldots, a_n, \ldots) \| = \| e^{i\theta_0} a_{\pi(0)}, \ldots, e^{i\theta_n} a_{\pi(n)}, \ldots) \|$ for every permutation π of N and every $(\theta_0, \ldots, \theta_n, \ldots)$ in $\mathbb{R}^N$,

(ii) $\| (1, 0, \ldots, 0, \ldots) \| = 1$,

(iii) either E is separable or it satisfies the Fatou property, namely $(a_0, \ldots, a_n, \ldots)$ lies in E as soon as $\sup_{N \geq 0} \| (a_0, \ldots, a_N, 0, \ldots \|)$ is finite, and $\| (a_0, \ldots, a_n, \ldots) \| = \sup_{N \geq 0} \| (a_0, \ldots a_N, 0, \ldots) \|$.

Note that if $(a_0, a_1, \ldots, a_n, \ldots) \in E$ and $((a_0, a_1, \ldots, a_N, 0, \ldots))_{N \geq 1}$ is not a Cauchy sequence in E, E has a closed subspace isomorphic to ℓ^∞, hence E is separable if and only if ℓ^1 is norm dense in E. If E is separable, E^* is a symmetric sequence space. If $E \neq \ell^\infty$, E embeds canonically in c_0 [S, Theorem 1.16]. If E satisfies the Fatou property, E is the dual space of a separable symmetric sequence space F, namely the norm closure of finitely supported sequences in E^*.

If E is a symmetric sequence space $E^{(p)}$ is also a symmetric sequence space ; E and $E^{(p)}$ are simultaneously separable or satisfy the Fatou property simultaneously.

Let H be a separable Hilbert space. $B(H)$ denotes the space of bounded linear operators on H, $K(H)$ denotes the space of compact operators. Excepted in part V, τ denotes the usual trace.

For $A \in B(H)$

$$| A |= (A^*A)^{1/2}$$

and for $A \in K(H)$ $(s_n(A))_{n\geq 0}$ is the sequence of eigenvalues of $| A |$, arranged in non increasing order, each one being counted as many times as its multiplicity order.

For every hermitian $A \in K(H)$ we denote by Q the hermitian projection on $\ker A$ and $P = \mathrm{Id}_H - Q$. We also denote by $\mathcal{P}(A)$ the set of spectral projections π of A such that $\pi A^{-1} \in B(H)$. Note that $P \in \mathcal{P}(A)$ if and only if the spectrum of $A \in K(H)$ is a finite set (in particular if H is finite dimensional). We denote by A^+ the positive part of A and $A^- = A^+ - A \geq 0$. Let $E \neq \ell^\infty$ be a symmetric sequence space. The Schatten space $C_E = C_E(B(H))$ is the space of compact operators A on H such that $(s_n(A))_{n\geq 0}$ lies in E, with

$$\| A \|_{C_E} = \| (s_n(A))_{n\geq 0} \|_E \, .$$

C_{ℓ^p} is denoted by C^p $(1 \leq p < \infty)$. $\mathrm{Re}\, C_E$ denotes the set of hermitian elements in C_E.

We recall that if $(P_n)_{n\geq 1}$ is an increasing sequence of hermitian projections: $H \to H$, if $P = \bigvee_{n\geq 1} P_n$ and if E is separable

$$(I.1) \qquad \forall x \in C_E \qquad \| Px - P_n x \|_{C_E} \to 0 \quad (n \to +\infty).$$

Indeed $(P_{n+1} - P_n)_{n\geq 1}$ is a w.u.c. sequence in $B(H)$, $(P_{n+1}x - P_n x)_{n\geq 1}$ is a w.u.c. sequence in C^1 if $x \in C^1$; (I.1) holds true in C^1 because C^1 is weakly complete ; as C^1 is norm dense in C_E (I.1) holds true in C_E.

An operator $B \in B(H)$ defines a continuous linear form on C_E if $\forall A \in C_E$, $< B, A >= \tau(B^*A)$ is finite. If E is separable and $E \neq \ell^1$ the dual space of C_E is C_{E^*} [S, Theorem 3.2], the dual space of C^1 is $B(H)$. For any symmetric sequence space E an element $\ell \in C_E^*$ is called positive (respectively hermitian) if it is a positive linear form on C_E (respectively $< \ell, A >$ is real for every $A \in \mathrm{Re}\, C_E$). The set of hermitian linear forms is denoted by $\mathrm{Re}\, C_E^*$. For $\ell \in C_E^*$ we define ℓ^* by

$$A \to < \overline{\ell, A^*} > .$$

For $R \in B(H)$ and $\ell \in C_E^*$ we denote

$$R\ell : C_E \to \mathbb{C} \quad A \to <\ell, R^*A>,$$
$$\ell R : C_E \to \mathbb{C} \quad A \to <\ell, AR^*>.$$

If $\ell \in \operatorname{Re} C_E^*, (R\ell)^* = \ell R^*$.

If E is separable, every $\ell \in C_E^*$ lies in $B(H)$ and the above definitions coincide with the usual ones for bounded operators.

II. Linear functionals supporting the unit ball of C_E at$A$

Definition II.1: Let X be a Banach space, $A, B \in X$, $A \neq 0$. Let

$$\mathcal{S}_A = \{\ell \in X^* \,|\, \|\ell\| = 1, \ <\ell, A> = \|A\|\},$$
$$G_A(B) = \lim_{t \to 0+} \frac{\|A + tB\| - \|A\|}{t}.$$

Note that the set $\mathcal{S}_A$ of norm one linear functionals supporting the unit ball of X at $A/\|A\|$ is a w^* compact convex subset of X^*. As the function $t \to \|A+tB\|$ is convex and continuous on $\mathbf{R}$ the limit in the definition of $G_A(B)$ actually exists.

We will study $G_A(B)$ when $X = C_E$ and compare $\mathcal{S}_A$ and $\mathcal{S}_{A^2}$ for $A \in C_{E^{(2)}}$.

Note that for $X = C^p$ $(1 < p < \infty)$ and $A \in C^p$, $\|A\|_{C^p} = 1$, $A \geq 0$, $\mathcal{S}_A = \{A^{p-1}\}$ because C^p is smooth, by Clarkson–McCarthy inequalities [S, Theorem 1.21], hence the results of this chapter are obvious for $C_E = C^p$.

We recall the following

Lemma II.2: *Let X be a Banach space, $A, B \in X$, $A \neq 0$. Then*

(i) $G_A(B) = \lim_{t \to 0+} \frac{\|A+tB\| - \|A\|}{t} = \sup_{\ell \in \mathcal{S}_A} Re <\ell, B>$

(ii) $\forall p > 1$, $\lim_{t \to 0+} \frac{\|A+tB\|^p - \|A\|^p}{t} = p \|A\|^{p-1} G_A(B)$.

Proof: For (i) see for example [DS, chapter V.9, Theorem 5, Lemma 10].

(ii) follows from (i) by the chain rule:

$$\lim_{t \to 0+} \frac{\|A+tB\|^p - \|A\|^p}{t} = \lim_{t \to 0+} \frac{\|A+tB\|^p - \|A\|^p}{\|A+tB\| - \|A\|} G_A(B)$$
$$= \lim_{y \to \|A\|} \frac{y^p - \|A\|^p}{y - \|A\|} G_A(B) = p \|A\|^{p-1} G_A(B).$$

The following lemma is an easy consequence of Lemma II.2 (i) and the definitions.

LEMMA II.3: *Let E be a symmetric sequence space and $A \neq 0$, $A \in C_E$. Let P be the projection on $(ker\, A)^{\perp}$ and $P + Q = Id$, let $Re\, \mathcal{S}_A = \{Re\, \ell \mid \ell \in \mathcal{S}_A\}$.*
(i) *Let $A \in Re\, C_E$. Then for every $\ell \in \mathcal{S}_A$, $P\ell P \in \mathcal{S}_A$ and $P\ell P + Q\ell Q \in \mathcal{S}_A$.*
(ii) *Let $A \in Re\, C_E$. Then $\ell^* \in \mathcal{S}_A$ for every $\ell \in \mathcal{S}_A$ and $Re\, \mathcal{S}_A \subset \mathcal{S}_A$.*
(iii) *Let $A \in Re\, C_E$. Then $Re\, \mathcal{S}_A = \mathcal{S}_A$ iff*

$$\forall B \in Re\, C_E \quad G_A(iB) = 0 \quad i.e. \; \| A + itB \| = \| A \| + o(t).$$

(iv) *Let $A \in Re\, C_E$. Then*

$$\forall B \in Re\, C_E \quad G_A(B) = \sup_{\ell \in Re\, \mathcal{S}_A} < \ell, B > .$$

(v) *Let $A = U \mid A \mid$ be a polar decomposition of A. Then*

$$\forall B \in C_E \quad G_A(B) = G_{|A|}(U^* B) \;\; and \;\; U^* \mathcal{S}_A = \mathcal{S}_{|A|}.$$

Proof: (i) Let $\ell \in \mathcal{S}_A$. Then, as $A = AP = PAP$

$$\| A \| = < \ell, A > = < \ell, PAP + QAQ > = < P\ell P, A > = < P\ell P + Q\ell Q, A > \leq \| A \|$$

by [S, Theorem 1.19].
(ii) Let $\ell \in \mathcal{S}_A$. Then

$$\| A \| = < \ell, A > = \overline{< \ell, A >} = < \ell^*, A >$$

hence $\ell^* \in \mathcal{S}_A$ and $\mathrm{Re}\, \ell = \frac{\ell + \ell^*}{2} \in \mathcal{S}_A$ by the convexity of $\mathcal{S}_A$.
(iii) If $A, B \in \mathrm{Re}\, C_E$, $G_A(iB) = G_A(-iB)$. Hence by Lemma II.2(i)

$$\sup_{\ell \in \mathcal{S}_A} \mathrm{Re} < \ell, iB > = - \inf_{\ell \in \mathcal{S}_A} \mathrm{Re} < \ell, iB > .$$

Hence $G_A(iB) = 0$ iff

$$\forall \ell \in \mathcal{S}_A, \quad \mathrm{Re} < \ell, iB > = 0, \quad \text{i.e. } < \ell, B > \in \mathbb{R}.$$

(iv) By Lemma II.2(i), if $B \in \mathrm{Re}\, C_E$

$$\begin{aligned} G_A(B) &= \sup_{\ell \in \mathcal{S}_A} \mathrm{Re} < \ell, B > = \sup_{\ell \in \mathcal{S}_A} < \mathrm{Re}\, \ell, B > \\ &= \sup_{\ell \in \mathrm{Re}\, \mathcal{S}_A} < \ell, B > \qquad \text{by (ii).} \end{aligned}$$

(v) Let $A = U \mid A \mid$. Then $\| A + tB \| = \| U \mid A \mid + tUU^* B \| = \| \mid A \mid + tU^* B \|$ and for every $\ell \in \mathcal{S}_A$

$$\| A \| = < \ell, A > = < \ell, U \mid A \mid > = < U^* \ell, \mid A \mid > = \| \mid A \mid \| .$$

LEMMA II.4: *Let E be a symmetric sequence space such that E^* has a strictly increasing norm. Let $A_0 \in Re\, C_E$ and let Q be the projection on $ker\, A$. Let $X = QXQ \in C_E$. Then*

$$\| A + tX \|_{C_E} = \| A \| + o(t)$$

and for every $\ell \in \mathcal{S}_A$, $Q\ell Q = 0$.

Proof: As $\| A + tX \| = \|| A + tX |\| = \|| A | + t | X |\|$ for $t \geq 0$ we may assume $A, X \geq 0$. As the ranges of the hermitian operators A, X are orthogonal, the set of eigenvalues of $A + tX$ is the union of the set $(s_n(A))_{n\geq 0}$ of eigenvalues of A and the set $(ts_n(X))_{n\geq 0}$ of eigenvalues of tX. Hence

$$\| A + tX \|_{C_E} = \| s(A) + ts(X) \|_E$$

where $s(A)$ and $s(X)$ are disjointly supported positive sequences whose decreasing rearrangement are $(s_n(A))_{n\geq 0}$ and $(s_n(X))_{n\geq 0}$ respectively. Let $\mathcal{A} \subset \mathbb{N}$ be the support of $s(A)$. Let $L \in \mathrm{Re}\,\mathcal{S}_{s(A)} \subset \mathcal{S}_{s(A)} \subset E^*$. Then L^+ and $1_{\mathcal{A}}L^+ \in E^*$ because E^* is a Banach lattice and L^+ belongs to $\mathcal{S}_{s(A)}$ because

$$\| s(A) \|_E = < L, s(A) > = < L^+ - L^-, s(A) > \leq < L^+, s(A) > \leq \| s(A) \|_E .$$

$1_{\mathcal{A}}L^+$ also belongs to $\mathcal{S}_{s(A)}$. As E^* has a strictly increasing norm, $1 = \| L^+ \| = \| 1_{\mathcal{A}}L^+ \|$ implies $L^+ = 1_{\mathcal{A}}L^+$. By Lemmas II.2, II.3,

$$G_{s(A)}(s(X)) = \sup_{L\in \mathrm{Re}\,\mathcal{S}_{s(A)}} < L, s(X) > = \sup_{L\in \mathrm{Re}\,\mathcal{S}_{s(A)}} < L^+, s(X) >$$

which proves the first claim because $< L^+, s(X) > = < 1_{\mathcal{A}}L^+, s(X) > = 0$. By the first assertion and Lemma II.2, for every $X \in C_E$ such that $QXQ = X$

$$\sup_{\ell\in\mathcal{S}_A} \mathrm{Re} < \ell, X > = 0$$

which implies $< \ell, X > = 0$ for every $\ell \in \mathcal{S}_A$.

LEMMA II.5: *Let E be a symmetric sequence space and let $A \geq 0$, $A \in C_E$, $A \neq 0$. Then*

(i) $\forall \ell \in \mathcal{S}_A$, $\forall \pi \in \mathcal{P}(A)$, $\pi\ell\pi \geq 0$.

(ii) *If E is separable*

$$\forall \ell \in \mathcal{S}_A, \qquad P\ell P \geq 0$$

where P is the projection on $(ker\, A)^{\perp}$.

Proof: (i) For every $B \in \operatorname{Re} C_E$ and $\ell \in \operatorname{Re} \mathcal{S}_A$, $t \in \mathbb{R}$

$$\| A + itB \| \geq |< \ell, A + itB >| = |\| A \| + it < \ell, B >| \geq \| A \| .$$

Let $R \in \operatorname{Re} B(H)$. Then

$$\begin{aligned} \| A + itA^{1/2}RA^{1/2} \|_{C_E} &= \| A^{1/2}(\mathrm{Id} + itR)A^{1/2} \|_{C_E} \\ &\leq \| A^{1/2} \|^2_{C_{E^{(2)}}} \| \mathrm{Id} + itR \|_{B(H)} \\ &= \| A \|_{C_E} (1 + t^2 \| R \|^2_{B(H)})^{1/2} \end{aligned}$$

hence $G_A(iA^{1/2}RA^{1/2}) = 0$; as in the proof of Lemma II.3(iii)

$$(II.1) \qquad \forall \ell \in \mathcal{S}_A, \qquad < \ell, A^{1/2}RA^{1/2} > \in \mathbb{R}.$$

On the other hand, for $R \geq 0$ and $0 \leq t < \| R \|^{-1}_{B(H)}$,

$$\begin{aligned} \| A - tA^{1/2}RA^{1/2} \|_{C_E} &= \| A^{1/2}(\mathrm{Id} - tR)A^{1/2} \|_{C_E} \\ &\leq \| A^{1/2} \|^2_{C_{E^{(2)}}} \| \mathrm{Id} - tR \|_{B(H)} \\ &\leq \| A \|_{C_E} \end{aligned}$$

hence $G_A(-A^{1/2}RA^{1/2}) \leq 0$; by (II.1) and Lemma II.2(i),

$$0 \geq \sup_{\ell \in \mathcal{S}_A} \operatorname{Re} < \ell, -A^{1/2}RA^{1/2} > = \sup_{\ell \in \mathcal{S}_A} - < \ell, A^{1/2}RA^{1/2} >$$

hence for every $\ell \in \mathcal{S}_A$, $< \ell, A^{1/2}RA^{1/2} > \geq 0$. Let now $\pi \in \mathcal{P}(A)$. Every $B \in \operatorname{Re} C_E$ such that $B = \pi B \pi$ can be written as

$$B = A^{1/2}(A^{-1/2}\pi B \pi A^{-1/2})A^{1/2} \quad \text{and} \quad R = A^{-1/2}\pi B \pi A^{-1/2} \in \operatorname{Re} B(H).$$

This proves the claim. (ii) is a consequence of (i) and (I.1).

LEMMA II.6: *Let E be a symmetric sequence space and let $A \neq 0$, $A \in C_{E^{(2)}}$, $B \in C_{E^{(2)}}$. Then*

$$\text{(i)} \qquad 2 \| A \| G_A(B) = G_{A^*A}(A^*B + B^*A).$$

$$\text{(ii)} \qquad \sup_{\ell \in \mathcal{S}_A} \| A \| \, Re < \ell, B > = \sup_{\ell \in Re\, \mathcal{S}_{A^*A}} Re < A\ell, B > .$$

(iii) $$\mathcal{S}_A = \frac{A}{\| A \|} Re\, \mathcal{S}_{A^*A} = \frac{A}{\| A \|} \mathcal{S}_{A^*A}.$$

(iv) *Let* $A \in Re\, C_{E^{(2)}}$, *P be the projection on* $(ker\, A)^\perp$. *Then*

$$\forall \ell \in \mathcal{S}_A, \qquad \ell = P\ell P.$$

Proof: (i) For $t \in \mathbb{R}$,

$$\| A+tB \|^2_{C_{E^{(2)}}} = \| (A^*+tB^*)(A+tB) \|_{C_E} = \| A^*A+t(A^*B+B^*A)+t^2B^*B \|_{C_E}.$$

The claim follows from Lemma II.2(ii).

(ii) The claim follows from (i) and Lemmas II.2(i), II.3(ii) because

$$\begin{aligned} G_{A^*A}(A^*B+B^*A) &= \sup_{\ell \in \mathcal{S}_{A^*A}} \mathrm{Re} < \ell, A^*B+B^*A > \\ &= \sup_{\ell \in \mathrm{Re}\, \mathcal{S}_{A^*A}} < \ell, A^*B+B^*A > \\ &= 2 \sup_{\ell \in \mathrm{Re}\, \mathcal{S}_{A^*A}} \mathrm{Re} < \ell, A^*B > \\ &= 2 \sup_{\ell \in \mathrm{Re}\, \mathcal{S}_{A^*A}} \mathrm{Re} < A\ell, B > . \end{aligned}$$

(iii) By the definitions and Lemma II.3(ii)

$$\frac{A}{\| A \|} \mathrm{Re}\, \mathcal{S}_{A^*A} \subset \frac{A}{\| A \|} \mathcal{S}_{A^*A} \subset \mathcal{S}_A.$$

All these sets are bounded, w^*-closed and convex in the dual space of $C_{E^{(2)}}$. By (ii) the polar sets of $\frac{A}{\|A\|}\mathrm{Re}\, \mathcal{S}_{A^*A}$ and $\mathcal{S}_A$ are the same in $C_{E^{(2)}}$; by the bipolar theorem $\mathcal{S}_A$ lies in the closed convex hull of $\frac{A}{\|A\|}\mathrm{Re}\, \mathcal{S}_{A^*A}$ and $\{0\}$, hence $\mathcal{S}_A = \frac{A}{\|A\|}\mathrm{Re}\, \mathcal{S}_{A^*A}$.

(iv) (iii) implies $\ell = P\ell$ for every $\ell \in \mathcal{S}_A$; by Lemma II.3(ii) $\ell^* \in \mathcal{S}_A$ hence $\ell^* = P\ell^*$, $\ell = \ell P$, which proves the claim.

Though the next results are not used in the proof of Theorem 0 we think they are interesting in themselves.

PROPOSITION II.7: *Let E be a separable symmetric sequence space, let* $A \neq 0$, $A \in Re\, C_E$. *Let P be the projection on* $(ker\, A)^\perp$ *and* $P+Q = Id$. *Then for every* $\ell \in \mathcal{S}_A$,

(i) $\ell = P\ell P + Q\ell Q$;

(ii) *if E^* has a strictly increasing norm $\ell = P\ell P$.*

If $A \geq 0$ or more generally if $A = A^+ - A^-$ where the spectra of A^+, A^- have at most $\{0\}$ in common

(iii) $P\ell P \in Re\,\mathcal{S}_A$

(iv) $A\ell = \ell A$.

Proof: Note that $\mathcal{S}_A \subset C_E^* \subset B(H)$. (i) implies (ii) by Lemma II.4.

(a) If (i) holds true for $A \geq 0$, it also holds for $A \in \mathrm{Re}\,C_E$: indeed for $A \in \mathrm{Re}\,C_E$ let $A = U \mid A \mid$ be a polar decomposition. Then $U \in \mathrm{Re}\,C_E$, U and $\mid A \mid$ commute in $B(H)$ hence U commutes with the projection P on $(ker\,A)^\perp = (ker \mid A \mid)^\perp$, by [R, Theorem 12.22]. By Lemma II.3(v), $\mathcal{S}_A = U\mathcal{S}_{|A|}$. As (i) holds true for every $\ell \in \mathcal{S}_{|A|}$ it also holds true for every $\ell \in \mathcal{S}_A$.

(b) We now prove (iii), (iv), (i) for $A \geq 0$: let $A \geq 0$, $\ell \in \mathcal{S}_A$, then $P\ell P \in \mathcal{S}_A$ by Lemma II.3(i) and $P\ell P \geq 0$ by Lemma II.5 which implies (iii). As $A \geq 0$, $A^{1/2} \in C_{E^{(2)}}$; for every $\ell \in \mathcal{S}_A$, $A^{1/2}\ell \in \mathcal{S}_{A^{1/2}}$; note that $\mathcal{S}_{A^{1/2}} \subset C^*_{E^{(2)}} \subset B(H)$. By Lemma II.6(iv) applied to $A^{1/2}$, $PA^{1/2}\ell P = A^{1/2}\ell$, and by Lemma II.5 applied to $A^{1/2}$, $PA^{1/2}\ell P \geq 0$. In particular $A^{1/2}\ell = \ell A^{1/2}$ which implies (iv). A and ℓ commute in $B(H)$ hence ℓ commutes with P and Q, $P\ell Q = Q\ell P = 0$, which implies (i).

(c) It remains to prove (iii) and (iv) when the spectra of A^+, A^- have at most $\{0\}$ in common. Let $A = U \mid A \mid$. Then $U = \varphi(\mid A \mid)$ where φ is a measurable function on the spectrum of $\mid A \mid$, with values in $\{+1, -1\}$. As (iv) holds true for $\mid A \mid$ every $\ell' \in \mathcal{S}_{|A|}$ satisfies $\mid A \mid \ell' = \ell' \mid A \mid$ hence $U\ell' = \ell' U$ by [R, 12.24]. As (iii) holds true for $\mid A \mid$, $P\ell' P \in \mathrm{Re}\,\mathcal{S}_{|A|}$. By Lemma II.3(v) every $\ell \in \mathcal{S}_A$ can be written as $\ell = U\ell'$, $\ell' \in \mathcal{S}_{|A|}$. As

$$PU\ell' P = P\ell' UP = P\ell' PU = UP\ell' P$$

(iii) is proved for A. As

$$A\ell = U \mid A \mid U\ell' = U \mid A \mid \ell' U = U\ell' \mid A \mid U = \ell A$$

(iv) is proved for A.

Remark II.8: Assertions (iii) and (iv) in the above proposition cannot be extended to all $A \in \mathrm{Re}\,C_E$ for $E = E^{(2)} = c_0$: let $H = \ell_2^2$, $C^\infty = C^\infty(B(\ell_2^2))$; let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$A \in \operatorname{Re} C^\infty$ and $\| A \|_{C^\infty} = 1$.

Let

$$\ell = 1/2 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 1/2i \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then

$$\ell^* \ell = 1/2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$\| \ell \|_{C^1} = 1$, $\ell \in \mathcal{S}_A$, $\ell \notin \operatorname{Re} \mathcal{S}_A$, $A\ell \neq \ell A$.

Also note that, for

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$| A + itB |^2 = (1 + t^2)\mathrm{Id} + 2it \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\forall t \in \mathbb{R}, \qquad \| A + itB \|^2_{C^\infty} = 1 + 2 \,| t |\, + t^2 \neq 1 + o(t)$$

which is not surprising in view of Lemma II.3(iii). This last example was considered in [GTJ, p. 184].

PROPOSITION II.9: *Let E be a separable symmetric sequence space. Let $1 < p < \infty$. Let $A \geq 0$, $A \in C_{E^{(p)}}$, $A \neq 0$. Then*

$$\mathcal{S}_A = \left(\frac{A}{\| A \|} \right)^{p-1} \mathcal{S}_{A^p}.$$

Proof: For $p = 2$, Lemma II.6(iii) implies the result. We now assume $\| A \|_{C_{E^{(p)}}} = 1$. Obviously $A^{p-1} \mathcal{S}_{A^p} \subset \mathcal{S}_A$ hence by the Hahn–Banach theorem we only have to prove that for every $B \in C_{E^{(p)}}$

$$(II.1) \qquad \sup_{\ell \in \mathcal{S}_A} \operatorname{Re} < \ell, B > = \sup_{\ell \in A^{p-1} \mathcal{S}_{A^p}} \operatorname{Re} < \ell, B > .$$

By Lemma II.5 and Proposition II.7, every $\ell \in \mathcal{S}_A$ satisfies (i) $\ell = P\ell P$, (ii) $\ell \geq 0$, (iii) $A\ell = \ell A$, where P is the projection on $(\ker A)^\perp$.

Let $A = \sum_{j \geq 0} \lambda_j P_j$ where the λ_j's are the distinct eigenvalues of A, counted according to their decreasing order for $j \geq 1$, $\lambda_0 = 0$, and the P_j's are the orthogonal projections on the corresponding eigenspaces. Then for $\ell \in \mathcal{S}_A$, $B \in C_{E^{(p)}}$,

$$\operatorname{Re} < \ell, B > = < \ell, \operatorname{Re} B > = < \ell, \sum_{j \geq 0} P_j \operatorname{Re} B \, P_j > = < \ell, \sum_{j \geq 1} P_j \operatorname{Re} B \, P_j >$$

because $\ell P_j = P_j \ell$ $(j \geq 0)$.

As $E^{(p)}$ is separable $\| \sum_{j\geq 1} P_j \mathrm{Re}\, B\, P_j - \sum_{j=1}^{N} P_j \mathrm{Re}\, B\, P_j \|_{C_{E(p)}} \to 0$ $(N \to +\infty)$ by (I.1). Hence it is enough to prove (II.1) for $B \in \mathrm{Re}\, C_{E(p)}$ such that $AB = BA$ and $\| A^{-1}B \|_{B(H)}$ is finite. Let $0 < t_0 < 1$ be such that

$$t_0^{1/4} \| A^{-1}B \|_{B(H)} < 1 \quad ; \quad \forall 0 \leq t \leq t_0, \qquad A + tB \geq 0.$$

(The problem is actually to show that $\mathcal{S}_A = A^{p-1}\mathcal{S}_{A^p}$ in E^*.) Let

$$(1+u)^p = \sum_{k\geq 0} \alpha_k u^k \qquad (| u |< 1).$$

For $0 \leq t \leq t_0$,

$$| A+tB |^p = (A+tB)^p = A^p(\mathrm{Id}+tA^{-1}B)^p = A^p + tpA^{p-1}B + A^p \sum_{k\geq 2} \alpha_k t^k (A^{-1}B)^k$$

hence

$$\begin{aligned} \lim_{t\to 0+} \frac{\|| A+tB |^p\|_{C_E} - \| A^p \|_{C_E}}{t} &= \lim_{t\to 0+} \frac{\| A^p + tpA^{p-1}B \|_{C_E} - \| A^p \|_{C_E}}{t} \\ &= \lim_{t\to 0+} \frac{\| A+tB \|^p_{C_{E(p)}} - \| A \|^p_{C_{E(p)}}}{t}. \end{aligned}$$

By Lemma II.2, $pG_A(B) = pG_{A^p}(A^{p-1}B)$, which proves (II.1) for such a B.

Comments on Part II:

(1) It has been shown in the proof of Lemma II.5 that for every $A \geq 0$ in C_E, every $\pi \in \mathcal{P}(A)$ and every $\ell \in \mathcal{S}_A$, $\pi\ell\pi \in \mathrm{Re}\, C_E^*$. Lemma II.6(iii) gives another proof of this fact: indeed $A^{1/2} \in C_{E(2)}$, $A^{1/2}\ell \in \mathcal{S}_{A^{1/2}}$ hence there exists $\ell_1 \in \mathrm{Re}\,\mathcal{S}_A$ such that $A^{1/2}\ell = A^{1/2}\ell_1$; it implies $\pi\ell = \pi\ell_1$, $\pi\ell\pi = \pi\ell_1\pi$ and $\pi\ell_1\pi \in \mathrm{Re}\, C_E^*$.

By Proposition II.7, it follows that $\ell \in \mathrm{Re}\, C_E^*$ if E is separable and p-convex for a $p > 1$. By Lemma II.3(iii) and [TJ] the separability assumption on E is not necessary: N. Tomczak proves that if E is p-convex $(p > 1)$, $A \geq 0$, $A, B \in \mathrm{Re}\, C_E$

$$\| A \|^p_{C_E} \leq \| A + itB \|^p_{C_E} \leq \| A \|^p_{C_E} + 2^p \,| t |^p \| B \|^p,$$

in particular

$$\| A + itB \|^p_{C_E} = \| A \|^p_{C_E} + o(t).$$

(2) Let E be separable, let $A \geq 0$ in $C_{E^{(2)}}$. Then Lemmas II.5, II.6(iii) and the Cauchy–Schwarz inequality imply that for every $\ell \in \mathcal{S}_A$,

$$\forall B \in \operatorname{Re} C_{E^{(2)}}, \quad |< \ell, B >| \leq < \ell', B^2 >^{1/2}$$

where $\ell = A\ell'$, $\ell' \in \mathcal{S}_{A^2}$. This implies Theorem 0 for the linear form $T = \ell : C_{E^{(2)}} \to \mathbb{C}$.

III. The study of $\| A + itB \|_{C_{E^{(2)}}}$ around $t = 0$ up to the second order and the set $I(A)$

We now study $\| A + itB \|_{C_{E^{(2)}}}$ around $t = 0$ up to the second order for $A \geq 0$ and B hermitian.

Note first that for $A \in \operatorname{Re} C_E$ and every $X \in C_E$ such that $X = QXQ$ (Q being the projection on $\ker A$)

$$\| A + itX \|^2_{C_{E^{(2)}}} = \| A^2 + t^2 X^2 \|_{C_E} .$$

We now consider operators B in $\operatorname{Re} C_E$ such that $QBQ = 0$. In view of the following key result we will consider operators B with a special form.

LEMMA III.1: *Let E be a symmetric sequence space, $A \in \operatorname{Re} C_E$, $R \in \operatorname{Re} B(H)$. Then for $t \in \mathbb{R}$,*

(i) $\| A \|_{C_E} = \| A + it(AR + RA) - t^2(RAR + \frac{R^2A + AR^2}{2}) \|_{C_E} + o(t^2)$.

(ii) $\| A + it(AR + RA) \|_{C_E} = \| A + t^2(RAR + \frac{R^2A + AR^2}{2}) \|_{C_E} + o(t^2)$.

Proof: This is an obvious consequence of the fact that

$$\| A \|_{C_E} = \| e^{itR} A e^{itR} \|_{C_E},$$

$$\| A + it(AR + RA) \|_{C_E} = \| e^{-itR}(A + it(AR + RA))e^{-itR} \|_{C_E}$$

and

$$\| e^{itR} - (\mathrm{Id} + itR - \frac{t^2}{2} R^2) \|_{B(H)} = o(t^2).$$

Hence it is natural to consider operators $B = AR + RA$. Using the results of part II we get the following technical lemma :

LEMMA III.2: *Let E be a symmetric sequence space and let $A \geq 0 \in C_{E^{(2)}}$. Let $\ell \in Re\,\mathcal{S}_A$ and let $\ell = A\ell'$ where $\ell' = P\ell' P \in Re\,\mathcal{S}_{A^2}$ as in lemma II.6. Then for every $R \in Re\,B(H)$ and every $\pi \in \mathcal{P}(A)$*

$$< \pi\ell\pi, RAR + \frac{R^2A + AR^2}{2} > \leq < \pi\ell'\pi, (AR + RA)^2 > .$$

This lemma is obvious in the C^{2p} case where $\mathcal{S}_A = \{A^{2p-1}\}$, $\mathcal{S}_{A^2} = \{A^{2p-2}\}$ for $A \geq 0$, $\| A \|_{C^{2p}} = 1$.

Proof: Lemma II.6 implies the existence of ℓ' ; as $\ell \in \mathrm{Re}\,\mathcal{S}_A$, $A\ell' = \ell' A$ (as linear forms on $C_{E^{(2)}}$) and $\ell = P\ell P$. For $\pi \in \mathcal{P}(A)$,

$$\begin{aligned} < \pi\ell'\pi, (AR + RA)^2 > &= < \pi\ell'\pi, ARAR + RARA + AR^2A + RA^2R > \\ &= 2 < \pi\ell\pi, RAR > + < \pi\ell\pi, \frac{R^2A + AR^2}{2} > \\ &\quad + < \pi\ell'\pi, RA^2R > . \end{aligned}$$

As $A \geq 0$, $\mathcal{P}(A) = \mathcal{P}(A^2)$ and by lemma II.5, $\pi\ell\pi \geq 0$, $\pi\ell'\pi \geq 0$. Hence $< \pi\ell\pi, RAR > \geq 0$ and $< \pi\ell'\pi, RA^2R > \geq 0$, which proves the claim.

The above lemmas motivate the following definitions :

Definition III.3: Let E be a symmetric sequence space and let $A \in Re\,C_E$. Let

$$I(A) = \{AR + RA \mid R \in Re\,B(H)\} \subset Re\,C_E.$$

For any projection p commuting with A

$$I(A,p) = \{B \in I(A) \mid B = pBp\} = \{AR + RA \mid R \in Re\,B(H),\ R = pRp\}.$$

These subsets are reasonably big in $Re\,C_E$, under suitable conditions on A :

LEMMA III.4: *Let E be a symmetric sequence space. Let $A \in Re\,C_E$ be such that the spectra of A^+, A^- have at most $\{0\}$ in common. Then*
(i) *if the spectrum of A is a finite set*

$$Re\,C_E = I(A) \oplus Q(Re\,C_E)Q,$$

(ii) *for every $\pi \in \mathcal{P}(A)$*

$$(\pi + Q)Re\,C_E(\pi + Q) = I(A, \pi + Q) \oplus Q(Re\,C_E)Q,$$

(iii) *if E is separable (respectively if $E = F^*$ where F is separable)*

$$\left\{ \bigcup_{\pi \in \mathcal{P}(A)} I(A, \pi + Q) \right\} \oplus Q(Re\, C_E)Q \quad \text{and} \quad I(A) \oplus Q(Re\, C_E)Q$$

are norm dense (respectively $\sigma(C_E, C_F)$ dense) in $Re\, C_E$.

Proof: (i) is a consequence of (ii) for $\pi = P$.

(ii) Let $A = \sum_{j \geq 0} \lambda_j P_j$ where the λ_j's are the distinct eigenvalues of A, the P_j's are the pairwise orthogonal eigenprojections of A, with $P_0 = Q$, $\sum_{j \geq 1} P_j = P$; for $j \geq 1$ the λ_j's are counted according to the decreasing order of the $| \lambda_j |$'s. For every $\pi \in \mathcal{P}(A)$ there exists $n \geq 1$ such that

$$\pi \leq \sum_{j=1}^{n} P_j.$$

Let $B \in \mathrm{Re}\, C_E$ be such that

$$B = (\pi + Q)B(\pi + Q) = \left(\sum_{j=0}^{n} P_j \right) B \left(\sum_{j=0}^{n} P_j \right) = \sum_{0 \leq i,\, j \leq n} \sum P_i B P_j.$$

The assumption on A implies $\lambda_i + \lambda_j \neq 0$ excepted if $i = j = 0$. Hence

$$B = AR + RA + QBQ$$

where

$$R = \sum_{\substack{0 \leq i, j \leq n \\ (i,j) \neq (0,0)}} \sum (\lambda_i + \lambda_j)^{-1} P_i B P_j$$

hence $R = (\pi + Q)R(\pi + Q)$.

(iii) Let $(\pi_n)_{n \geq 1}$ be an increasing sequence in $\mathcal{P}(A)$ such that $P = \bigvee_n \pi_n$. Then by (I.1) $\bigcup_n \{(\pi_n + Q)\mathrm{Re}\, C_E(\pi_n + Q)\}$ is norm dense in C_E if E is separable (respectively $\sigma(C_E, C_F)$ dense if $E = F^*$ and F is separable). Hence (ii) implies (iii).

We will use the technical Lemma III.2 in the proof of Theorem 0. However the following proposition is more significant and will give a more natural proof in the smooth case (see Lemma IV.3) :

PROPOSITION III.5: *Let E be a symmetric sequence space and let $A \geq 0 \in C_{E^{(2)}}$ be such that the spectrum of A is a finite set. Then*

$$\forall B \in Re\, C_{E^{(2)}}, \qquad \| A + itB \|^2_{C_{E^{(2)}}} \leq \| A^2 + 2t^2B^2 \|_{C_E} + o(t^2).$$

Proof: Let $B \in \operatorname{Re} C_{E^{(2)}}$. By Lemma III.4(i), $B = AR + RA + X$ where $R \in \operatorname{Re} B(H)$ and $X = QXQ \in \operatorname{Re} C_{E^{(2)}}$. As in Lemma III.1 we get
$\| A + itB \|^2_{C_{E^{(2)}}}$

$$\begin{aligned}
&= \| A + it(AR + RA) + itX \|^2_{C_{E^{(2)}}} \\
&= \| e^{-itR}(A + it(AR + RA) + itX)e^{-itR} \|^2_{C_{E^{(2)}}} \\
&= \| A + itX + t^2(RX + XR + RAR + \frac{R^2A + AR^2}{2}) \|^2_{C_{E^{(2)}}} + o(t^2) \\
&= \| A^2 + t^2(X^2 + 2\operatorname{Re} A(RX + XR + RAR + \frac{R^2A + AR^2}{2}) \|_{C_E} + o(t^2) \\
&= \| A^2 \|_{C_E} \\
&\quad + t^2 \sup_{\ell' \in \operatorname{Re} \mathcal{S}_{A^2}} \operatorname{Re} < \ell', X^2 + 2A(RX + XR + RAR + \frac{R^2A + AR^2}{2}) > \\
&\quad + o(t^2) \\
&= \| A \|^2_{C_{E^{(2)}}} + o(t).
\end{aligned}$$

By Lemma II.3, $\operatorname{Re} \mathcal{S}_A = \mathcal{S}_A$; as $A\ell' \in \mathcal{S}_A$ for $\ell' \in \operatorname{Re} \mathcal{S}_{A^2}$ we get $A\ell' = \ell' A = PA\ell' P$ for $\ell' \in \operatorname{Re} \mathcal{S}_{A^2}$, in particular

$$< \ell', A(RX + XR) >= 0$$

and by Lemma III.2 applied to $\pi = P \in \mathcal{P}(A)$ and $\ell = A\ell'$
$\| A + itB \|^2_{C_{E^{(2)}}}$

$$\begin{aligned}
&\leq \| A^2 \|_{C_E} + t^2 \sup_{\ell' \in \operatorname{Re} \mathcal{S}_{A^2}} < \ell', X^2 + 2P(RA + AR)^2 P > + o(t^2) \\
&= \| A^2 \|_{C_E} + t^2 \sup_{\ell' \in \operatorname{Re} \mathcal{S}_{A^2}} < P\ell' P + Q\ell' Q, X^2 + 2(RA + AR)^2 > + o(t^2) \\
&\leq \| A^2 \|_{C_E} + 2t^2 \sup_{\ell' \in \operatorname{Re} \mathcal{S}_{A^2}} < P\ell' P + Q\ell' Q, X^2 + (RA + AR)^2 > + o(t^2) \\
&= \| A^2 \|_{C_E} + 2t^2 \sup_{\ell' \in \operatorname{Re} \mathcal{S}_{A^2}} < P\ell' P + Q\ell' Q, (RA + AR + X)^2 > + o(t^2) \\
&\leq \| A^2 + 2t^2B^2 \|_{C_E} + o(t^2)
\end{aligned}$$

because, by Lemma II.3, for every $\ell' \in \operatorname{Re} \mathcal{S}_{A^2}$, $P\ell' P$ and $P\ell' P + Q\ell' Q \in \operatorname{Re} \mathcal{S}_{A^2}$.

Remark III.6: The factor 2 in proposition III.5 is the best possible, as it is shown by the following example [GTJ]: let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A^2 = P, \quad B = \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}.$$

Then

$$| A + itB |^2 = \begin{pmatrix} 1 + t^2 \mid b \mid^2 & itb \\ -it\bar{b} & t^2 \mid b \mid^2 \end{pmatrix},$$

$$\| A + itB \|^2_{C^\infty} = \frac{1}{2} + t^2 \mid b \mid^2 + \frac{1}{2}\sqrt{1 + 4t^2 \mid b \mid^2} = 1 + 2t^2 \mid b \mid^2 + o(t^2),$$

$$\| A^2 + 2t^2 B^2 \|_{C^\infty} = 1 + 2t^2 \mid b \mid^2 .$$

Note that $B = AB + BA$; $\mathcal{S}_A = \mathcal{S}_{A^2} = \{A\} = \{P\}$. With the notation of Lemma III.2

$$B = R, \quad RAR = RA^2R = \begin{pmatrix} 0 & 0 \\ 0 & \mid b \mid^2 \end{pmatrix}, \quad < \ell, RAR > = < A, RAR > = 0,$$

$$< \ell', RA^2R > = < A, RAR > = 0.$$

Comments on Part III:

(1) In the finite dimensional case another proof of Proposition III.5 (or a variant of it for $\| A + itB - t^2 C \|_{C_{E^{(2)}}}, C \in C_{E^{(2)}}$) goes as follows (this was our original proof): let $A \in \mathrm{Re}\, C_E$, let $M(t) = | A + itB |^2 = A^2 + it(AB - BA) + t^2 B^2$; by perturbation theory [K, Chapter II.6] the eigenvalues $(\lambda_j(t))_{j=0}^{j=N-1}$ of the hermitian operator $M(t)$ $(t \in \mathbb{R})$ are analytic around $t = 0$ and there exists an analytic determination of a basis of orthonormal eigenvectors $(v_j(t))_{j=0}^{N-1}$ around $t = 0$. This does not imply that the decreasing rearrangement $(s_j(t))_{j=0}^{j=N-1}$ of $(\lambda_j(t))_{j=0}^{j=N-1}$ is analytic around $t = 0$, a counterexample can be found in Remark II.8, where $s_0(t)$ is not analytic.

We denote by $(\lambda_j)_{j=0}^{N-1}$ the eigenvalues of A.

The equation

$$M(t) v_j(t) = \lambda_j(t) v_j(t)$$

allows the computation of an order 2 expansion of $\lambda_j(t)$ around $t = 0$. For $A \geq 0$, using the fact that $v_j(0)$ is both an eigenvector for $(A^2, \lambda_j(0) = \lambda_j^2)$ and for (A, λ_j) one gets $\lambda_j(t) = \lambda_j(0) + bt^2 + o(t^2)$. Using the fact that

$$-1 \leq \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \leq 1$$

for distinct eigenvalues λ_i, λ_j of A and the computation of b one gets

$$0 \leq \lambda_j(t) \leq < A^2 + 2t^2 B^2 \ , \ v_j(0) \otimes v_j(0) > +o(t^2).$$

Hence for $0 \leq n \leq N-1$

$$\sum_0^n s_j(t) = \sum_{j \in J_t} \lambda_j(t) \leq \sum_{j \in J_t} < A^2 + 2t^2 B^2, v_j(0) \otimes v_j(0) > +o(t^2)$$

where $J_t \subset \{0, \dots, N-1\}$ and cardinal $J_t = n+1$. Hence for every $\epsilon > 0$

$$\sum_0^n s_j(t) \leq \sum_0^n s_j(A^2 + 2t^2 B^2) + o(t^2) \leq \sum_0^n s_j(A^2 + 2t^2 B^2 + t^2 \epsilon \mathrm{Id})$$

for t small enough, and by standard arguments [S, Theorem 1.9]

$$\||\, A + itB \,|^2\|_{C_E} \leq \|\, A^2 + 2t^2 B^2 \,\|_{C_E} + o(t^2).$$

(2) In view of Proposition III.5 and Tomczak inequality (2) [TJ, Proposition 1] the following question is natural: does there exists a constant $K \geq 2$ such that for $A \geq 0$, $A, B \in \operatorname{Re} C_{E(2)}$

$$\|\, A + iB \,\|^2_{C_{E(2)}} \leq \|\, A^2 + KB^2 \,\|_{C_E} \ ?$$

We can give a positive answer in $C^\infty = K(H)$ and in C^4 with $K=2$, in C^6 with $K = \sqrt{5}$ ($K = 2$ if $B \geq 0$). We sketch the proofs. In C^∞ we must prove

$$s_0(A^2 + i(AB - BA) + B^2) \leq s_0(A^2 + 2B^2).$$

By the proof of [TJ, Proposition 1] it is enough to prove it for $H = \ell_2^2$ hence for

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad 0 \leq \lambda_2 \leq \lambda_1.$$

In this case the above eigenvalues can be exactly computed, the comparison is not so easy but it can be done.

In C^4 we get (τ denotes the usual trace)

$$\|\, A + iB \,\|^4_{C^4} = \tau((A^2 + B^2)^2) + 2\tau(A^2 B^2) - 2\tau((AB)^2).$$

As $A \geq 0$

$$\tau((AB)^2) = \tau(A^{1/2}BABA^{1/2}) \geq 0,$$

hence

$$\| A + iB \|_{C^4}^4 \leq \tau((A^2 + B^2)^2) + 2\tau((A^2 + B^2)B^2) + \tau(B^4) = \| A^2 + 2B^2 \|_{C^2}^2 .$$

The computation in C^6 is more complicated but follows the same idea. It uses the fact that for $C = A^{1/2}BA^{1/2}$,

$$0 \leq \| AC - CA \|_2^2 = 2(\| A^{3/2}BA^{1/2} \|_2^2 - \| ABA \|_2^2)$$

hence

$$-\tau(A^2(AB - BA)^2) = \| A(AB - BA) \|_2^2$$

$$= \| A^2B \|_2^2 + \| ABA \|_2^2 - 2 \| A^{3/2}BA^{1/2} \|_2^2 \leq \tau(A^4B^2).$$

IV. The factorization theorem for operators: $C_E^{(2)} \to \mathcal{H}$

$\mathcal{H}$ denotes a Hilbert space.

Definition IV.1: Let E be a symmetric sequence space and let $T: C_{E^{(2)}} \to \mathcal{H}$ be a bounded linear operator. T is K-factorizable if there exists a bounded linear form f on C_E such that $\| f \| = 1$ and

$$\forall x \in C_{E^{(2)}} \quad \| T(x) \| \leq K \| T \| < f, \frac{x^*x + xx^*}{2} >^{1/2} .$$

Note that f is necessarily a positive linear form. The aim of this part is to prove Theorem 0 which we rewrite as follows :

THEOREM IV.2: *Let E be a symmetric sequence space such that E^* has a strictly increasing norm. Then every bounded linear operator T: $C_{E^{(2)}} \to \mathcal{H}$ is $2\sqrt{2}$ factorizable.*

We first give the core of the proof, in the finite dimensional case (Corollary IV.4). The reduction steps are standard and will be given afterwards in Lemmas IV.6, IV.7.

LEMMA IV.3: *Let E be a symmetric sequence space such that E^* has a strictly increasing norm. Let T: $C_{E^{(2)}} \to \mathcal{H}$ be a norm one operator attaining its norm at $A \geq 0$, $\| A \|_{C_{E^{(2)}}} = 1$. Let P be the projection on $(ker\ A)^\perp$, $P + Q = Id$. Then*

(i) *For every $X \in Re\, C_{E^{(2)}}$ such that $X = QXQ$, $T(X) = 0$.*

(ii) *Let $\ell' \in Re\, \mathcal{S}_{A^2}$ be such that $\ell' = P\ell' P$ and $A\ell' = Re\ T^*T(A)$, which is possible by lemma II.6. Then for every $\pi \in \mathcal{P}(A)$ and for every $B \in I(A, \pi + Q)$*

$$\| T(B) \|^2 \leq 2 < \ell', B^2 > . \qquad (IV.1)$$

(iii) *If the spectrum of A is a finite set or if E is separable, (IV.1) holds true for every $B \in Re\, C_{E^{(2)}}$.*

Proof: Note that

$$1 = \| T(A) \|^2 = < T^*T(A), A > = < Re\, T^*T(A), A >$$

hence $\mathrm{Re}\, T^*T(A) \in \mathrm{Re}\, \mathcal{S}_A$. Let $B \in \mathrm{Re}\, C_{E^{(2)}}$ and let ϵ be a random variable such that $P(\epsilon = +1) = P(\epsilon = -1) = 1/2$. Then

$$(IV.2) \qquad \begin{cases} \| T(A) \|^2 + t^2 \| T(B) \|^2 = E \| T(A) + \epsilon i t T(B) \|^2 \\ \qquad \leq E \| A + \epsilon i t B \|^2_{C_{E^{(2)}}} = \| A + itB \|^2_{C_{E^{(2)}}} . \end{cases}$$

We first mention a transparent proof of (iii) when E is smooth and the spectrum of $A \geq 0$ is a finite set: by Proposition III.5, for $B \in Re\, C_{E^{(2)}}$

$$\begin{aligned} \| A + itB \|^2_{C_{E^{(2)}}} &\leq \| A^2 + 2t^2 B^2 \|_{C_E} + o(t^2) \\ &= 1 + 2t^2 < \ell', B^2 > + o(t^2) \end{aligned}$$

where $\mathrm{Re}\, \mathcal{S}_{A^2} = \mathcal{S}_{A^2} = \{\ell'\}$ by the smoothness assumption on E (by [A], C_E is smooth if and only if E is smooth).

As $\| T(A) \|^2 = 1$, (IV.2) implies $\| T(B) \|^2 \leq 2 < \ell', B^2 >$.

We now give a general proof when E is not assumed to be smooth.

(i) Let $X = QXQ \in \mathrm{Re}\, C_{E^{(2)}}$. By Lemma II.4,

$$\| A + itX \|^2_{C_{E^{(2)}}} = \| A^2 + t^2 X^2 \|_{C_E} = 1 + o(t^2)$$

hence (IV.2) implies $T(X) = 0$.

(ii) Let $B = AR + RA \in I(A)$, and let

$$S = RAR + \frac{R^2A + AR^2}{2}.$$

By Lemma III.1,

$$\| A - t^2S + it(AR + RA) \|^2_{C_{E^{(2)}}} = 1 + o(t^2)$$

hence by (IV.2) applied to $A - t^2S$ and $AR + RA$

$$\| T(A - t^2S) \|^2 + t^2 \| T(AR + RA) \|^2 \leq 1 + o(t^2)$$

$$\| T(A) \|^2 - 2t^2 \mathrm{Re} < T(A), T(S) > + t^2 \| T(AR + RA) \|^2 + o(t^2) \leq 1 + o(t^2)$$

$$\| T(AR + RA) \|^2 \leq 2\mathrm{Re} < T^*T(A), S > = 2 < \mathrm{Re}\, T^*T(A), S > .$$

By Lemma III.2 applied to $\ell = \mathrm{Re}\, T^*T(A) \in \mathrm{Re}\, S_A$

$$\| T(AR + RA) \|^2 \leq 2 < \ell', (AR + RA)^2 >$$

for every $AR+RA \in I(A, \pi+Q)$ and every $\pi \in \mathcal{P}(A)$. (Note that $(\pi+Q)\ell(\pi+Q) = \pi\ell\pi$).

(iii) Lemma III.4 and (i), (ii) imply (IV.1) for every $B \in \mathrm{Re}\, C_{E^{(2)}}$ such that $B = (\pi + Q)B(\pi + Q)$ for a $\pi \in \mathcal{P}(A)$. This ends the proof if $P \in \mathcal{P}(A)$, i.e. if the spectrum of A is a finite set. If E is separable, by Lemma III.4 again, (IV.1) holds true for a norm dense subset of $\mathrm{Re}\, C_{E^{(2)}}$ hence for every $B \in \mathrm{Re}\, C_{E^{(2)}}$: indeed

$$(IV.3) \qquad \begin{aligned} \| B^2 - B_n^2 \|_{C_E} &\leq \| B^2 - BB_n \|_{C_E} + \| BB_n - B_n^2 \|_{C_E} \\ &\leq \left(\| B \|_{C_{E^{(2)}}} + \| B_n \|_{C_{E^{(2)}}} \right) \| B - B_n \|_{C_{E^{(2)}}} \end{aligned}$$

by [S, Theorem 2.8], hence $\| B^2 - B_n^2 \|_{C_E} \to 0$ if $\| B - B_n \|_{C_{E^{(2)}}} \to 0$ $(n \to \infty)$.

COROLLARY IV.4: *Let E be a symmetric sequence space such that E^* has a strictly increasing norm. For $n \geq 1$, let H_n be a Hilbert space with dimension n. Let T: $C_{E^{(2)}}(B(H_n)) \to \mathcal{H}$ be a bounded linear operator. Then T is $2\sqrt{2}$-factorizable.*

Proof: We may assume $\| T \| = 1$. As $C_{E^{(2)}}(B(H_n))$ is finite dimensional T attains its norm at A, $\| A \|_{C_{E^{(2)}}} = 1$ and the spectrum of A is a finite set, hence Lemma IV.3 and the subsequent Lemma IV.5 imply the claim.

By Lemma IV.3 the statement of the corollary still holds true for an infinite dimensional separable Hilbert space H if E is separable and $E^{(2)}$ reflexive, which implies that $C_{E^{(2)}}(B(H))$ is reflexive.

LEMMA IV.5 ([H, Appendix]): *Let E be a symmetric sequence space and let $K > 0$. Let T: $C_{E^{(2)}} \to \mathcal{H}$ be a bounded linear operator which attains its norm at $A \in C_{E^{(2)}}$, $\| A \| = 1$. Let $A = U_0 \mid A \mid$ be a polar decomposition of A. We assume that there exists $f \in C_E^*$ with $\| f \| = 1$ such that*

$$\forall x \in \operatorname{Re} C_{E^{(2)}}, \qquad \| T(U_0 x) \| \leq K \| T \| < f, x^2 >^{1/2} .$$

Then the operator $x \to T(U_0 x)$ is $\sqrt{2}K$-factorizable and T is $2K$ factorizable.

Proof: Let $y \in C_{E^{(2)}}$, then $y = \operatorname{Re} y + i \operatorname{Im} y$ and

$$(\operatorname{Re} y)^2 + (\operatorname{Im} y)^2 = 1/2(y^* y + y y^*).$$

By assumption

$$\begin{aligned} \| T(U_0 y) \| &\leq \| T(U_0 \operatorname{Re} y) \| + \| T(U_0 \operatorname{Im} y) \| \\ &\leq \sqrt{2}(\| T(U_0 \operatorname{Re} y) \|^2 + \| T(U_0 \operatorname{Im} y) \|^2)^{1/2} \\ &\leq \sqrt{2}K \| T \| < f, \frac{y^* y + y y^*}{2} >^{1/2} . \end{aligned}$$

Let $x \in C_{E^{(2)}}$, hence $x = U_0 y$ where $y = U_0^* x$, and

$$\begin{aligned} \| T(x) \| &\leq \sqrt{2}K \| T \| < \frac{f + U_0 f U_0^*}{2}, x^* x + x x^* >^{1/2} \\ &\leq 2K \| T \| < \frac{\varphi}{\| \varphi \|}, \frac{x^* x + x x^*}{2} >^{1/2} \end{aligned}$$

where

$$\varphi = \frac{f + U_0 f U_0^*}{2}.$$

LEMMA IV.6: *Let E be a symmetric sequence space. The set of K-factorizable operators: $C_{E^{(2)}} \to \mathcal{H}$ with norm less than one is closed for the topology of strong convergence.*

Proof: Let (T_α) be a net of operators: $C_{E^{(2)}} \to \mathcal{H}$, strongly converging to T, such that $\| T \| = 1$, $\| T_\alpha \| \leq 1$ and such that there exists (f_α), satisfying

(i) $\| f_\alpha \|_{C_E^*} = 1$;

(ii) $\forall \alpha$, $\forall x \in C_{E^{(2)}}$, $\| T_\alpha(x) \| \leq K \| T_\alpha \| < f_\alpha, \frac{x^* x + x x^*}{2} >^{1/2}$.

Let f be a w^*-limit of (f_α). Then

$$\| T(x) \| = \lim_\alpha \| T_\alpha(x) \| \leq K < f, \frac{x^* x + x x^*}{2} >^{1/2} .$$

LEMMA IV.7: *Let E be a symmetric sequence space. Then every norm one operator $T: C_E \to \mathcal{H}$ lies in the strong closure of the set $\mathcal{F}_E$ of operators $t: C_E \to \mathcal{H}$ of norm less than one for which there exists a hermitian projection $P: H \to H$ with finite dimensional range H_n such that*

$$\forall x \in C_E, \qquad t(x) = t(PxP).$$

Such a $t \in \mathcal{F}_E$ can be identified with an operator: $C_E(B(H_n)) \to \mathcal{H}$.

Proof: Let $(P_n)_{n\geq 1}$ be an increasing sequence of hermitian projections: $H \to H$ with finite dimensional ranges such that $\bigvee_n P_n = \mathrm{Id}_H$. For every $T: C_E \to \mathcal{H}$ let T_n be defined by

$$\forall x \in C_E, \qquad T_n(x) = T(P_n x P_n).$$

If E is separable $\| x - P_n x P_n \|_{C_E} \to 0$ $(n \to +\infty)$ for every $x \in C_E$ by (I.1) hence $(T_n)_{n\geq 1}$ strongly converges to T, and $\| T_n \| \leq \| T \|$.

If E is not separable, $E = F^*$ where F is a separable symmetric sequence space, and $C_E = C_F^*$. The space $B(C_F^* \to \mathcal{H})$ of bounded linear operators: $C_F^* \to \mathcal{H}$ is the dual space of the projective tensor product $C_F^* \hat{\otimes} \mathcal{H}$ and the bidual of the space $C_F \hat{\hat{\otimes}} \mathcal{H}$ of compact operators: $C_F^* \to \mathcal{H}$. Hence the unit ball of $B(C_F^* \to \mathcal{H})$ is the w^*-closure (and the strong closure) of the unit ball of $C_F \hat{\hat{\otimes}} \mathcal{H}$. The last unit ball is the norm closure of norm less than one operators $t: C_E \to \mathcal{H}$ such that t^*: $\mathcal{H} \to C_F$ has a finite dimensional range. For such a t there exist finite sequences $(z_i)_{i\leq k} \in C_F$ and $(e_i)_{i\leq k} \in \mathcal{H}$ such that

$$\forall x \in C_E, \qquad t(x) = \sum_{i=1}^{k} < z_i, x > e_i.$$

As F is separable, $\| z_i - P_n z_i P_n \|_{C_F} \to 0$ $(n \to \infty,\ i \leq k)$ by (I.1), hence

$$\| t - t_n \|_{B(C_E \to \mathcal{H})} \to 0 \quad (n \to \infty).$$

Hence the unit ball of $C_F \hat{\hat{\otimes}} \mathcal{H}$ is the norm closure of its intersection with $\mathcal{F}_E$, which ends the proof.

As we already mentioned, Corollary IV.4, Lemmas IV.6 and IV.7 imply Theorem IV.2.

Comments on Part IV:
In the statement of Theorem IV.2 the Hilbert space $\mathcal{H}$ can be replaced by any Banach space X whose modulus of uniform convexity satisfies $\delta_X(\epsilon) \geq K\epsilon^2$ $(\epsilon > 0)$, if moreover $C_{E^{(2)}}$ or X^* has the λ-bounded approximation property. The factorization constant then depends on K and λ. The first assumption on X implies that X is reflexive [LT, Proposition 1.e.3]. The approximation property assumption ensures the validity of Lemma IV.7 with X instead of $\mathcal{H}$ [DU]. By the assumption on X,

$$\forall x, y \in X, \qquad \| x \|^2 + K' \| y \|^2 \leq E \| x + \epsilon y \|^2$$

where K' is a constant depending on K [B, part V, Chap. I.3, Lemma 2]. This implies an obvious modification in (IV.2), namely $\| T(A) \|^2 + K't^2 \| T(B) \|^2 \leq E \| T(A) + \epsilon i t T(B) \|^2$ and the rest of the proof of Lemma IV.3 is the same.

However as X has cotype 2 [LT, Theorem 1.e.16] this generalized statement is also a consequence of Theorem IV.2, of [P2, Theorem 4.1] and of [TJ, Theorem 1]: indeed $E^{(2)}$ has a dual or predual space which is a 2-concave symmetric sequence space, hence by [TJ] $C^*_{E^{(2)}}$ has cotype 2 ; by [P2] every T: $C_{E^{(2)}} \to X$, with a finite dimensional range, factors through a Hilbert space.

V. Generalization to $L_E(M,\tau)$ spaces

The aim of this part is to prove Theorem V.5 below which is a version of Theorem 0 for $L_E(M,\tau)$ spaces.

We first give definitions and properties of $L_E(M,\tau)$ spaces. Then we give the analogues of the main results of parts II and III: most proofs can be transcribed, except for Lemma III.4 (see Lemma V.4). Then we state Theorem V.5 and prove it in a particular case, we end by reducing the general case to this particular one.

Definition V.0: Let $\Omega = [0,1]$ or $[0,\infty[$ equipped with the σ-algebra of Borel sets and the Lebesgue measure. A symmetric function space E is a Banach lattice of functions on Ω such that

$$L^\infty[0,1] \subset E \subset L^1[0,1] \text{ if } \Omega = [0,1],$$

$$L^\infty[0,\infty] \cap L^1[0,\infty[\subset E \subset L^\infty[0,\infty[+ L^1[0,\infty[\text{ if } \Omega = [0,\infty[,$$

and such that equidistributed functions have the same norm in E. Moreover it is assumed either that E is σ-order continuous or E has the Fatou property, i.e.

$$(1) \qquad f_n \uparrow f \text{ a.s.}; \quad \sup_n \| f_n \|_E < \infty \Rightarrow f \in E, \quad \| f \|_E = \lim_n \| f_n \|_E .$$

Note that E and $E^{(2)}$ are simultaneously symmetric function spaces ; they are σ-order continuous or have the Fatou property simultaneously.

This setting is the analogue of symmetric sequence spaces (a symmetric sequence space is separable if and only if it is σ-order continuous). However a symmetric function space with the Fatou property is not a dual space in general (e.g. $E = L^1$) (though this is true for a symmetric sequence space) ; if moreover E is p-convex ($p > 1$) E is a dual space (see Lemma V.6).

Also note that if E is σ-order continuous E^* is a symmetric function space [LT, p. 29].

Definition V.0 is slightly more restrictive than the definition of a rearrangement invariant space in [LT, Definition 2a1]. By the discussion in [LT, p. 118] the two definitions agree on $[0,1]$ and Definition V.0 only excludes the case where E is a r. i. space on $[0,\infty[$ which is not isomorphic to $L^\infty[0,\infty[$ though $1_{[0,1]}E = L^\infty[0\ 1]$ isomorphically.

In this chapter (M,τ) will be a semifinite von Neumann algebra M of operators on a Hilbert space H, equipped with a faithful semi-finite normal trace τ on M ([T, V.2, Definition 2.1 and Theorem 2.15]). Let $\bar{M}$ denote the space of measurable operators on H with respect to (M,τ), i.e. (see e.g. [FK, Definition 1.2]) the space of densely defined closed operators A affiliated with M such that

$$\tau(\mathcal{E}_{(\lambda,+\infty)}) \to 0 \quad \text{as} \quad \lambda \to +\infty$$

where $\mathcal{E}$ is the spectral measure of $\mid A \mid$. Note that $\bar{M}$ is the closure of M with respect to the measure topology (for which a fundamental system of neighborhoods of 0 is, for $\epsilon, \delta > 0$,

$$V(\epsilon,\delta) = \{A \in \bar{M} \mid \exists P = P^2 \in M \qquad \| AP \|_M \leq \epsilon, \ \ \tau(\mathrm{Id} - P) \leq \delta\}).$$

Let $A \in \bar{M}$. The s^{th} singular number of A is (see e.g. [FK, Definition 2.1, Proposition 2.2, Lemma 2.5)

$$\mu_s(A) = \inf\{\| AP \|_M \mid P = P^2, \ \ \tau(\mathrm{Id} - P) \leq s\}, \quad s > 0$$

hence $\mu_s(A) = \mu_s(A^*) = \mu_s(\mid A \mid)$. If $A \geq 0$ and $\mathcal{E}$ denotes the spectral measure of A,

$$\mu_s(A) = \inf\{x \geq 0 \mid \tau(\mathcal{E}_{(x,\infty)}) \leq s\}.$$

We denote by $\mu(A)$ the function $]0,\infty[\to \mathbf{R}$, $s \to \mu_s(A)$.

Definition V.1: Let E be a symmetric function space and let (M,τ) be as above. The symmetric space $L_E(M,\tau)$ is the space of operators $A \in \bar{M}$ such that $\mu(A)$ lies in E and

$$\| A \|_{L_E(M,\tau)} = \| \mu(A) \|_E .$$

$L_E(M,\tau)$ is a Banach space (see e.g. [X1]).

Note that if $M = B(H)$, if τ is the usual trace then $M = \bar{M}$; if moreover E is a symmetric sequence space $L_E(M,\tau) = C_E$. Also note that $L_{L^\infty}(M,\tau) = M$; $L_{L^p}(M,\tau)$ is denoted by $L^p(M,\tau)$ for $1 \le p < \infty$.

An operator $A \in M$ has a τ-finite support if $\tau(P) < +\infty$ where P is the support projection of A, i.e. the smallest projection P such that $AP = A$.

If $\tau(\mathrm{Id}) = a < \infty$, then for every $A \in M$, $\mu(A)$ is a function on $[0,a]$. Then

$$M \subset L_E(M,\tau) \subset L^1(M,\tau)$$

hence $L_E(M,\tau)$ is norm dense in $L^1(M,\tau)$. In general

$$M \cap L^1(M,\tau) \subset L_E(M,\tau) \subset M + L^1(M,\tau).$$

If E is σ-order continuous, then $M \cap L^1(M,\tau)$ is norm dense in $L_E(M,\tau)$ by [X1, Lemma 4.5]. If moreover $\tau(\mathrm{Id})$ is finite, M is norm dense in $L_E(M,\tau)$.

Hermitian and positive operators are well defined in $\bar{M}$, hence in $L_E(M,\tau)$. Hermitian and positive bounded linear forms on $L_E(M,\tau)$ are defined by duality; if $\ell \in L_E^*(M,\tau)$ and $R \in M$, $R\ell$ and ℓR are also defined by duality as in the C_E case. If $A \in M$ has a τ finite support it defines a bounded linear form on $L_E(M,\tau)$ by

$$\forall B \in L_E(M,\tau), \qquad < A, B >= \tau(A^* B).$$

Let E be a σ-order continuous symmetric function space ; we know that E^* is a symmetric function space. It is not known in general if $L_{E^*}(M,\tau)$ is the dual space of $L_E(M,\tau)$. This is true if $\tau(\mathrm{Id})$ is finite by [X3, Lemma 1].

The following lemma gives the analogue of (I.1):

LEMMA V.2: *Let E be a symmetric function space, let (M,τ) be such that $\tau(Id)$ is finite. Let $(P_n)_{n\ge1}$ be an increasing sequence of hermitian projections in M and let $P = \bigvee_{n\ge1} P_n$. Then*

(a) *If E is σ-order continuous*

$$\forall x \in L_E(M,\tau), \quad \| Px - P_n x \|_{L_E} \to 0 \quad (n \to +\infty).$$

(b) *If F is a σ-order continuous symmetric function space and $E = F^*$,*

$$\forall x \in L_E(M,\tau), \quad Px - P_n x \xrightarrow[n\to+\infty]{} 0 \quad for \quad \sigma(L_E, L_F).$$

Proof: (a) The claim is proved in ([X3], Lemma 2) for $x = \mathrm{Id}$, i.e. $\| P - P_n \|_{L_E} \to 0$. Hence it is proved for $x \in M$ because

$$\forall x \in M, \quad \| Px - P_n x \|_{L_E} \leq \| x \|_M \| P - P_n \|_{L_E} .$$

As E is σ-order continuous and $\tau(Id) < +\infty$, M is norm dense in $L_E(M,\tau)$, which implies the claim because

$$\| P_n \|_M = \| P \|_M = 1.$$

(b) is proved by duality from (a) because $L_E(M,\tau) = L_F^*(M,\tau)$.

Definition V.3: Let $A \in Re\,\bar{M}$. We denote by $\mathcal{P}(A)$ the set of hermitian projections $\pi = \mathcal{E}(\mathcal{B})$ where $\mathcal{E}$ is the spectral measure of A and $\mathcal{B}$ is any Borel subset of $\mathbb{R}$, such that πA and πA^{-1} lie in M. Let P be the support projection of A, $P + Q = Id$.

Note that $P = \vee\{\pi \mid \pi \in \mathcal{P}(A)\}$.

We recall that $A\pi = \pi A$ ([R, Theorem 13.33]).

The following lemma extends Lemma III.4. $I(A)$ and $I(A,p)$ are defined as in Definition III.3, $L_E(M,\tau)$ replacing C_E and M replacing $B(H)$. We refer to [R, Theorems 13.30 and 13.24] for the symbolic calculus.

LEMMA V.4: *Let E be a symmetric function space, τ a finite trace on M. Let $A \in Re\,L_E(M,\tau)$ be positive (or more generally be such that $A = U \mid A \mid$ where $U = \varphi(\mid A \mid)$, φ being a continuous function on the spectrum of $\mid A \mid$, with values in $\{+1,-1\}$). Then*

(i) *for every $\pi \in \mathcal{P}(A)$, $I(A, \pi + Q)$ is norm dense (for the L_E norm) in*

$$\{B \in Re\,M \mid QBQ = 0,\ (\pi + Q)B(\pi + Q) = B\}.$$

(ii) *If E is σ-order continuous $\{\bigcup_{\pi\in\mathcal{P}(A)} I(A, \pi + Q)\} \oplus QRe\,L_E Q$ and $I(A) \oplus QRe\,L_E Q$ are norm dense in $Re\,L_E$.*

Proof: (i) Let X_E be the closure of $Re\,M$ in the L_E norm. We denote by X'_E the set of continuous $\mathbb{R}$-linear forms on the real space X_E. Let $\ell \in X'_E$ be such that

(1) $Q\ell Q = 0$; (2) $(\pi+Q)\ell(\pi+Q) = \ell$; (3) $\forall R \in \mathrm{Re}\,M\ < \ell, AR+RA >= 0$.

We must show that $\ell = 0$.

As $(\pi + Q)A \in \operatorname{Re} M$, (2) and (3) imply

$$\forall R \in \operatorname{Re} M, \quad < \ell, AR + RA > = < (\pi A)\ell + \ell(A\pi), R > = 0$$

hence in X'_E,

$$(V.1) \qquad \pi A\ell + \ell A\pi = 0$$

which implies

$$\pi A^2 \ell = \ell A^2 \pi$$

because $\pi A(\pi A\ell) = -\pi A(\ell A\pi) = (-\pi A\ell)A\pi = (\ell A\pi)A\pi$. The assumption on A implies that $\pi A = g(\pi A^2)$ where g is a continuous function on the spectrum of πA^2. As every continuous function on a bounded closed subset of $\mathbb{R}$ is a uniform limit of polynomials, πA is the norm limit in M of a sequence of polynomials $\mathcal{P}_n(\pi A^2)$. Hence $\mathcal{P}_n(\pi A^2)\ell = \ell\mathcal{P}_n(\pi A^2)$ and

$$(V.2) \qquad \pi A\ell = \ell\pi A = \ell A\pi.$$

By (V.1) and (V.2),

$$(V.3) \qquad \pi A\ell = 0 = \ell A\pi.$$

As $A^{-1}\pi \in \operatorname{Re} M$, (V.3) implies

$$\pi\ell = 0 = \ell\pi$$

hence by (1) and (2), $\ell = 0$.

(ii) As E is σ-order continuous and τ is finite, X_E is the whole of $\operatorname{Re} L_E(M,\tau)$ and by lemma V.2, $\bigcup_{\pi\in\mathcal{P}(A)}(\pi + Q)X_E(\pi + Q)$ is norm dense in X_E. Hence (i) implies (ii).

Let us observe that Lemmas III.1, II.4, II.6 remain valid in the setting of $L_E(M,\tau)$ spaces. The same proof as in Lemma II.5 shows that for $A \geq 0 \in L_E(M,\tau)$

$$(V.4) \qquad \forall \ell \in \mathcal{S}_A,\ \forall \pi \in \mathcal{P}(A),\ \forall B \geq 0,\ B \in M, \quad < \pi\ell\pi, B > \geq 0.$$

We want to prove

THEOREM V.5: *Let E be a p-convex symmetric function space ($1 < p < \infty$). Then every bounded linear operator $T: L_{E^{(2)}}(M,\tau) \to \mathcal{H}$ is $2\sqrt{2}$-factorizable.*

We recall that, if E is p-convex, the norm of E^* is strictly increasing.

If E (hence $E^{(2)}$) is σ-order continuous, if τ is finite, and if $L_{E^{(2)}}(M,\tau)$ is reflexive, then the same proof as in Corollary IV.4 implies Theorem V.5. But we cannot reduce Theorem V.5 to this particular case as we did for C_E spaces. Actually we will reduce Theorem V.5 to the particular case where τ is finite, E and $E^{(2)}$ are the dual spaces of the σ-order continuous function spaces E_* and $E_*^{(2)}$ and T^*: $\mathcal{H} \to M$ is compact. We first prove Theorem V.5 in this case. We need a technical lemma:

LEMMA V.6: *Let τ be a finite trace on M. We assume that E and $E^{(2)}$ are the dual spaces of the σ-order continuous symmetric function spaces E_* and $E_*^{(2)}$. Let $A \geq 0 \in L_{E^{(2)}}(M,\tau)$, let $(\pi_n)_{n\geq 1} \in \mathcal{P}(A)$ be an increasing sequence such that $P = \bigvee_n \pi_n$ is the support projection of A. Let $\ell \in Re\, S_A$ such that $\ell \in M$. Then $\ell \geq 0$ and there exists $\ell' \in Re\, S_{A^2}$ such that*

(a) $\ell = A\ell'$;
(b) $\forall n \geq 1 \;\; \pi_n \ell' \pi_n \in L_{E_*}(M,\tau)$
(c) $\ell' = \lim_n \pi_n \ell' \pi_n$ *for* $\sigma(L_E^*, L_E)$
(d) $\ell' \geq 0$.

Proof: As τ is finite, L_E is the dual space of L_{E_*} and $L_{E^{(2)}}$ is the dual space of $L_{E_*^{(2)}}$.

As $\ell \in M \subset L_{E_*^{(2)}}$ Lemma V.2 implies that $P\ell$ is the norm limit of $(\pi_n \ell)_{n\geq 1}$ in $L_{E_*^{(2)}}$.

By the analogue of Lemma II.6, there exists $\ell' \in \mathrm{Re}\, S_{A^2}$ such that $\ell = A\ell'$. Hence $\ell = P\ell = \ell P$ and

$$\forall n \geq 1, \qquad A^{-1}\pi_n \ell = \pi_n A^{-1}\ell = \pi_n \ell' \in M.$$

As $A\ell' = \ell' A$, $A^2\ell' = \ell' A^2 = A\ell = \ell A$, hence, as $\ell \in M$, $\pi_n \ell = \ell \pi_n$ [R, Theorem 13.13]. This implies $\pi_n \ell = \pi_n \ell \pi_n$ hence $\pi_n \ell' = \pi_n \ell' \pi_n \in \mathrm{Re}\, M$. As $(\pi_{n+1} - \pi_n)_{n\geq 1}$ is a w.u.c. series in M, $(\pi_n \ell')_{n\geq 1}$ is a weak Cauchy sequence in L_E^*, hence in L_{E_*}. Let $\lambda \in \mathrm{Re}\, L_E^*$ be its limit. Then $\pi_n \lambda = \pi_n \ell'$ ($n \geq 1$), the sequence $(A\pi_n \lambda)_{n\geq 1} = (\pi_n \ell)_{n\geq 1}$ is norm convergent to ℓ in $L_{E_*^{(2)}}$ and convergent to $A\lambda$ for $\sigma(L_{E^{(2)}}^*, L_{E^{(2)}})$, hence $A\ell' = A\lambda$. As $\| \lambda \|_{L_E^*} \leq 1$, $\lambda \in \mathrm{Re}\, S_{A^2}$. By (V.4),

$\pi_n \ell \pi_n \geq 0$ in $L^1(M,\tau)$ and in M, hence $\ell \geq 0$. By (V.4) again $\pi_n \lambda \pi_n \geq 0$ $(n \geq 1)$ hence $\lambda \geq 0$. λ satisfies conditions a,b,c,d, which proves the lemma.

Let us note that if E is p-convex $(1 < p \leq 2)$ E_* is q-concave $(\frac{1}{p} + \frac{1}{q} = 1)$ hence L_{E_*} has cotype q by [X2], the w.u.c. series $((\pi_{n+1} - \pi_n)\ell')_{n\geq 1}$ is unconditionally convergent in L_{E_*} and $\lambda \in L_{E_*}$.

LEMMA V.7: *Let τ be a finite trace on M, let E, $E^{(2)}$ be the dual spaces of σ-order continuous symmetric function spaces, such that E^* has a strictly increasing norm. Then every compact linear operator T: $L_{E^{(2)}}(M,\tau) \to \mathcal{H}$ such that $T^* : \mathcal{H} \to M$ is $2\sqrt{2}$ factorizable.*

Proof: By the analogue of Lemma IV.5 we may assume $\| T \| = 1 = \| T(A) \| = \| A \|_{L_{E^{(2)}}}$, $A \geq 0$. Let $\ell = \mathrm{Re}\, T^*T(A)$, hence $\ell \in \mathrm{Re}\, \mathcal{S}_A$, $\ell \in M$. By the same proof as in Lemma IV.3, if P is the support projection of A and $P + Q = \mathrm{Id}$,

(a) $\forall X = QXQ \in L_{E^{(2)}}, \quad T(X) = 0$

(b) $\forall n \geq 1, \quad \forall B \in I(A, \pi_n + Q)$

$$\| T(B) \|^2 \leq 2 < \ell', B^2 > \tag{V.5}$$

where $(\pi_n)_{n\geq 1}$ and ℓ' satisfy the conditions stated in Lemma V.6.

By the analogue of (IV.3), (V.5) still holds true for every B in the norm closure of $I(A, \pi_n + Q)$ in $L_{E^{(2)}}$, hence by a) and lemma V.4, (V.5) holds true for every B in $\mathrm{Re}\, M$ such that $B = (\pi_n + Q)B(\pi_n + Q)$ $(n \geq 1)$. We claim that (V.5) holds true for every $B \in \mathrm{Re}\, M$. Indeed let $B \in \mathrm{Re}\, M$, let $B_n = (\pi_n + Q)B(\pi_n + Q)$ $(n \geq 1)$. By Lemma V.2,

$$B_n \to B, \quad \sigma(L_{E^{(2)}}, L_{E_*^{(2)}}).$$

By assumption T is w^*-norm continuous, hence

$$\begin{aligned} \| T(B) \|^2 &= \lim \| T(B_n) \|^2 \leq 2 \liminf < \ell', B_n^2 > \\ &\leq 2 \liminf < \ell', (\pi_n + Q)B^2(\pi_n + Q) > \text{ because } \ell' \geq 0 \\ &= 2 \liminf < \ell', \pi_n B^2 \pi_n > \text{ because } Q\ell' = \ell' Q = 0 \\ &= 2 \lim < \pi_n \ell' \pi_n, B^2 > = < \ell', B^2 > \text{ by Lemma V.6.} \end{aligned}$$

We now claim that (V.5) holds true for every $B \in \mathrm{Re}\, L_{E^{(2)}}$; by the analogue of Lemma IV.5, it will imply the lemma. Indeed for every $B \in \mathrm{Re}\, L_{E^{(2)}}$ let $(p_n)_{n\geq 1}$

be an increasing sequence of projections in $\mathcal{P}(B)$ such that $\bigvee_n p_n$ is the support projection of B. Then $(p_nB)^2 \leq B^2$ and $p_nB \in M$ $(n \geq 1)$; by Lemma V.2

$$p_nB \to B, \quad \sigma(L_{E^{(2)}}, L_{E_*^{(2)}}).$$

Hence as $\ell' \geq 0$, $(V.5)$ applied to p_nB $(n \geq 1)$ implies

$$\| T(B) \| = \lim \| T(p_nB) \| \leq 2 \liminf < \ell', (p_nB)^2 > \leq 2 < \ell', B^2 > .$$

We now consider the reduction steps. The p-convexity assumption on E is used in the following lemma:

LEMMA V.8: *Let E be a p-convex symmetric function space $(p > 1)$. Then either*

(i) *$E = E_*^*$ where E_* is a σ-order continuous symmetric function space and*

$$L_E(M, \tau) = L_{E_*}^*(M, \tau),$$

or

(ii) *E and E^* are σ-order continuous and $L_{E^*}^*(M,\tau) = L_{E^{**}}(M,\tau)$. If τ is finite,*

$$L_{E^{**}}(M, \tau) = L_E^{**}(M, \tau).$$

E and $E^{(2)}$ satisfy (i) or (ii) simultaneously.

Proof: As E is p-convex E^* is a q-concave Banach lattice $(\frac{1}{p} + \frac{1}{q} = 1)$ hence E^* has no subspace isomorphic to c_0. By [LT, Proposition 1.a.7] E^* is σ-order continuous hence a symmetric function space. By Definition V.0, either E is σ-order continuous or E has the Fatou property. In this case by [LT, pp. 29–30] E is the dual space of E' (the set of integrals in E^*, see [LT, p. 29]): indeed $E = E''$; as E' is a closed subspace of E^*, E' is σ-order continuous, hence $E'' = E'^*$.

Note that $E^{(2)}$ is 2-convex ; E and $E^{(2)}$ are simultaneously σ-order continuous or satisfy simultaneously the Fatou property, hence are dual spaces simultaneously by the above proof.

If τ is finite the remaining assertions come from [X3, Lemma 1]. We claim that for a q-concave symmetric function space $L_{F^*}(M,\tau) = L_F^*(M,\tau)$ even if τ

is not finite. Indeed by the proof of [X3, Lemma 1] $L_{F^*} \subset L_F^*$ and we only have to show that $L_F^*(M,\tau) \subset \bar{M}$. By [LT, Proposition 2.b.3]

$$\forall q' > q, \quad L^1(0,\infty) \cap L^{q'}(0,\infty) \subset F$$

hence

$$L^1(M,\tau) \cap L^{q'}(M,\tau) \subset L_F(M,\tau).$$

Moreover $L^1(M,\tau) \cap L^{q'}(M,\tau)$ is norm dense in $L_F(M,\tau)$ by the σ-order continuity of F and [X1, Lemma 4.5]. As $L^{q'}(M,\tau)^* = L^{p'}(M,\tau)$ by [D] ($\frac{1}{q'} + \frac{1}{p'} = 1$)

$$L_F^*(M,\tau) \subset M + L^{p'}(M,\tau) \subset \bar{M}.$$

This proves the claim and ends the proof of the lemma.

The proof of Theorem V.5: By Lemma V.8 only two cases must be considered, either $E^{(2)}$ is σ-order continuous or $E^{(2)} = F^*$ where F is σ-order continuous. Moreover by the same proof as in Lemma IV.6, the set of $2\sqrt{2}$ factorizable operators with norm less than one is closed for the strong convergence topology.

If $E^{(2)}$ is σ-order continuous, by Lemma V.2 and the same proof as in Lemma IV.7, we may assume that $\tau(\mathrm{Id})$ is finite ; in this situation, by Lemma V.8,

$$L_{E^{(2)}}^{**}(M,\tau) = L_{E^{(2)**}}(M,\tau).$$

As every T: $L_{E^{(2)}} \to \mathcal{H}$ extends as T^{**}: $L_{E^{(2)}}^{**} \to \mathcal{H}$ with $\| T^{**} \| = \| T \|$ it is enough to prove the theorem for $L_{E^{(2)**}}(M,\tau)$, hence in the next setting.

Let $E^{(2)} = F^*$ where F is σ-order continuous. By Lemma V.8, $L_{E^{(2)}} = L_F^*$ hence by the same proof as in Lemma IV.7, it is enough to prove the theorem when $\tau(\mathrm{Id})$ is finite and T is such that T^*: $\mathcal{H} \to L_F(M,\tau)$ has a finite dimensional range. As M is norm dense in $L_F(M,\tau)$ we may even assume that T^*: $\mathcal{H} \to M$. By Lemma V.8 again, $E = E_*^*$ where E_* is a σ-order continuous symmetric function space, hence the assumptions of Lemma V.7 are satisfied. By this lemma such T's are $2\sqrt{2}$-factorizable, which ends the proof of the theorem.

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